Change-point problems for multivariate time series using pseudo-observations

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Brief review on change-points I

Inference methods for detecting change-points in the distribution of observations focused first on :

- (1-d, independent) detecting a change in the mean (Page, 1955), location parameters (Wolfe and Schechtman, 1984) and distribution functions (Csörgő and Horváth, 1987, Carlstein, 1988).
- Horváth (1993), Berkes et al. (2004) studied change-points in the parameters of the conditional mean and conditional variance (Górecki et al., 2018) of univariate time series.
- Change-point in the copula for multivariate independent random vectors was studied by Quessy et al. (2013), and Holmes et al. (2013) for the joint cdf and the copula.

Brief review on change-points II

- Extended by Bücher et al. (2014) who considered also alpha-mixing time series; they also proposed a new sequential empirical process specially designed for copula problems.
- Wied et al. (2012) worked with stationary alpha-mixing sequences to detect changes in their dependence structure using conditional correlation.
- Bai (1994), Ghoudi and Rémillard (2015) considered change-points in the innovations.

Here the focus is on change-point tests for non-observable independent random vectors ε_t associated with a multivariate time series $\mathbf{X}_t$.

Brief review on change-points III

- One could consider generalised error models (Du, 2016), i.e., for each fixed $j \in \{1, \ldots, d\}$, $\varepsilon_{t,j} = H_{\boldsymbol{\theta},t,j}(X_{t,j})$ are iid with a continuous distributions function F_j, where $H_{\boldsymbol{\theta},t,j}$ is a $\mathcal{F}_{t-1}$-measurable mapping and $\mathcal{F}_t$ is a sigma-algebra for which $\mathbf{X}_t$ is measurable.

- These models include stochastic volatility models, ARMA models, and also regime-switching models.

- In the latter case, $H_{\boldsymbol{\theta},t,j}(x_j) = P(X_{tj} \leq x_j | \mathcal{F}_{t-1})$, so F_j is the uniform distribution function over $(0, 1)$.

- Note that even if the univariate series $\varepsilon_{t,j}$ are iid for each fixed $j \in \{1, \ldots, d\}$, the cdf K_t of $\boldsymbol{\varepsilon}_t$ might depend on t (Nasri and Rémillard, 2019).

Brief review on change-points IV

Under the null hypothesis of no change-point, the ε_t are iid with continuous distribution function K, margins $F_1, \ldots, F_d$, and associated copula C. In fact, we want to test

$$\mathcal{H}_0 : \varepsilon_t \sim K, \quad t = 1, \ldots, n,$$
$$\text{vs}$$
$$\mathcal{H}_1 : \varepsilon_t \sim K, \quad t < \tau, \quad \varepsilon_t \sim K', \quad t = \tau, \ldots, n,$$

where $K \neq K'$ are continuous distribution functions on $\mathbb{R}^d$. One can also consider more than one change-point.

- In our context, the test statistics are functionals of sequential empirical processes of pseudo-observations $\mathbf{e}_{n,t}$, which are estimations of ε_t, since the latter are not observable in general.

Brief review on change-points V

- Performing change-point tests for $d > 1$ is more challenging than in the univariate case where in general the limiting distribution of the test statistics are distribution free.
- In fact, under the null hypothesis, the limiting distribution of these statistics depend on the unknown joint distribution function K or its unknown associated copula C, making it impossible to compute P-values (if not using Khmaladze transform!)
- To circumvent this problem, multipliers bootstrap and traditional bootstrap will be used.
- It will be shown that these bootstrapping procedures are valid for pseudo-observations.

Brief review on change-points VI

In this work, we do not consider a formal procedure for determining the number of change-points, as in Pein and Shah (2021), using cross-validation with absolute errors instead of squared errors.

Here are some interesting recent topics that are not covered in this talk:

- Detection of multiple change-points in drift parameters of mean-reverting processes (Nkurunziza and Fu, 2019)
- Minimax rates for a change-point in the mean for high-dimensional iid and weakly dependent random vectors (Liu et al., 2021)
- Detection of change-point in the mean for high-dimensional data with missing observations (Follain et al., 2021)

Change point tests I

Problem: we want to perform change-point tests for independent random vectors $\varepsilon_1, \ldots, \varepsilon_n$ that might be non-observable.

We assume that there exist pseudo-observations $\mathbf{e}_{n,t}$ estimating ε_t, $t \in \{1, \ldots, n\}$.

For example, for a generalised error model, one might take $e_{n,t,j} = H_{\boldsymbol{\theta}_n,t,j}(X_{t,j})$, where $\boldsymbol{\theta}_n$ is a consistent estimator of $\boldsymbol{\theta}$.

Finally, assume that $\mathbf{e}_{n,t}$ and ε_t take values in some closed rectangle T of $[-\infty, \infty]^d$.

Change point tests II

To perform the change-point tests, define

$$\mathbb{K}_n(s, \mathbf{y}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \{\mathbb{I}(\mathbf{e}_{n,t} \le \mathbf{y}) - K(\mathbf{y})\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function,

$$\alpha_n(s, \mathbf{y}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \{\mathbb{I}(\varepsilon_t \le \mathbf{y}) - K(\mathbf{y})\}, \quad (s, \mathbf{y}) \in [0, 1] \times T,$$

$$\beta_n(s, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \{\mathbb{I}(\mathbf{U}_t \le \mathbf{u}) - C(\mathbf{y})\}, \quad (s, \mathbf{u}) \in [0, 1]^{1+d},$$

where $\mathbf{U}_t = \mathbf{F}(\varepsilon_t)$, $t \in \{1, \ldots, n\}$, with $\mathbf{F}(\mathbf{y}) = (F_1(y_1), \ldots, F_d(y_d))$, and where the copula C is uniquely defined by $C \circ \mathbf{F}(\mathbf{y}) = K(\mathbf{y})$, $\mathbf{y} \in T$.

Change point tests III

Then, under $\mathcal{H}_0$, $(\alpha_n, \beta_n) \rightsquigarrow (\alpha, \beta)$, where α is a K-Kiefer process, i.e., a continuous centred Gaussian process with covariance function

$$\min(s,t)\left[\min\{K(\mathbf{y}), K(\mathbf{x})\} - K(\mathbf{y})K(\mathbf{x})\right], \quad s,t \in [0,1], \mathbf{x}, \mathbf{y} \in T.$$

β is a C-Kiefer process, and $\alpha(s, \mathbf{y}) = \beta\{s, \mathbf{F}(\mathbf{y})\}$.

Here $\rightsquigarrow$ means weak convergence in the Skorohod topology.

Note that for pseudo-observations, $\mathbb{K}_n$ does not converge to a Kiefer process (Ghoudi and Rémillard, 2004, Nasri and Rémillard, 2019).

Change point tests IV

Tests statistics for the change-point of the joint distribution function are often functionals of the process

$$\begin{aligned} \mathbb{A}_n(s, \mathbf{y}) &= \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \{\mathbb{I}(\mathbf{e}_{n,t} \leq \mathbf{y}) - K_n(\mathbf{y})\} \\ &= \mathbb{K}_n(s, \mathbf{y}) - \lambda_n(s)\mathbb{K}_n(1, \mathbf{y}), \end{aligned} \tag{1}$$

where K_n is the empirical cdf of the pseudo-observations and $\lambda_n(s) = \frac{\lfloor ns \rfloor}{n}$, $s \in [0, 1]$. Statistics based on $\mathbb{A}_n$:

$$\mathcal{T}_n = \max_{1 \leq j \leq n} \max_{1 \leq t \leq n} \left| \mathbb{A}_n (j/n, \mathbf{e}_{n,t}) \right|, \tag{2}$$

$$\mathcal{S}_n = \max_{1 \leq j \leq n} \sum_{t=1}^{n} \mathbb{A}_n^2(j/n, \mathbf{e}_{n,t}) \Big/ n. \tag{3}$$

Change point tests V

For the copula, one simply replaces the cdf by the empirical cdf to get

$$\begin{aligned} \mathbb{B}_n(s,\mathbf{u}) &= \sqrt{n}\,\lambda_n(s)\left\{C_n(s,\mathbf{u}) - C_n(1,\mathbf{u})\right\} \\ &= \mathbb{C}_n(s,\mathbf{u}) - \lambda_n(s)\mathbb{C}_n(1,\mathbf{u}), \end{aligned} \tag{4}$$

where $C_n(s,\mathbf{u}) = \frac{1}{\lfloor ns \rfloor}\sum_{t=1}^{\lfloor ns \rfloor} \mathbb{I}\{\mathbf{F}_n(1,\mathbf{e}_{n,t}) \le \mathbf{u}\}$ and

$$\mathbb{C}_n(s,\mathbf{u}) = \frac{1}{\sqrt{n}}\sum_{t=1}^{\lfloor ns \rfloor}\left[\mathbb{I}\{\mathbf{F}_n(1,\mathbf{e}_{n,t}) \le \mathbf{u}\} - C(\mathbf{u})\right].$$

Change point tests VI

To be able to work with pseudo-observations, one needs the following assumption:

Assumption 1

As $n \to \infty$, $\mathbb{K}_n \rightsquigarrow \mathbb{K}$, *where* $\mathbb{K}(s, \mathbf{y}) = \alpha(s, \mathbf{y}) - s\gamma(\mathbf{y})$, $(s, \mathbf{y}) \in [0,1] \times T$, *for some continuous stochastic process* γ.

Note that in the special case where the $\mathbf{X}_t$s are iid, then $\varepsilon_t = \mathbf{X}_t$ is observable and $\mathbf{e}_{n,t} = \varepsilon_t$, $t \in \{1, \ldots, n\}$, so $\gamma \equiv 0$.

In general, γ is the "price to pay" for not observation the ε_t.

It was shown in Nasri and Remillard (2019) that Assumption 1 is met for Generalised Error models.

Change point tests VII

When working with copulas, we will also assume the following condition (Segers, 2012):

Assumption 2 (Segers' condition)

For each $j \in \{1, \ldots, d\}$, the partial derivative $\partial_{u_j} C$ exists and is continuous on $\left\{\mathbf{u} \in [0,1]^d : 0 < u_j < 1\right\}$.

Change point tests VIII

The following result shows that the limiting distributions of $\mathbb{A}_n$ and $\mathbb{B}_n$ do not depend on nuisance parameters, i.e., they converge to the same processes as if the innovations ε_t were observed.

Lemma 1

Suppose that Assumption 1 holds. Then

(L1) $\mathbb{A}_n \rightsquigarrow \mathbb{A}$, *where* $\mathbb{A}(s, \mathbf{y}) = \alpha(s, \mathbf{y}) - s\alpha(1, \mathbf{y})$ *and* $\mathbb{A}(s, \mathbf{y}) = \mathbb{B}\{s, \mathbf{F}(\mathbf{y})\}$, *where* $\mathbb{B}(s, \mathbf{u}) = \beta(s, \mathbf{u}) - s\beta(1, \mathbf{u})$.

(L2) *If* C *satisfies Assumption 2, then* $\mathbb{B}_n \rightsquigarrow \mathbb{B}$.

In fact, from Assumption 1,

$$\begin{aligned}\mathbb{A}(s, \mathbf{y}) &= \mathbb{K}(s, \mathbf{y}) - s\mathbb{K}(1, \mathbf{y}) = \alpha(s, \mathbf{y}) - s\gamma(\mathbf{y}) - s\{\alpha(1, \mathbf{y}) - \gamma(\mathbf{y})\} \\ &= \alpha(s, \mathbf{y}) - s\alpha(1, \mathbf{y}).\end{aligned}$$

Change point tests IX

Remark 1

Statistics $\mathcal{T}_n$ and $\mathcal{S}_n$ are really a function of the ranks of the pseudo-observations, they converge in law to $\mathcal{T} = \sup_{(s,\mathbf{u})\in[0,1]^{1+d}} |\mathbb{B}(s,\mathbf{u})|$ *and* $\mathcal{S} = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{B}(s,\mathbf{u})\}^2 \, \mathrm{d}C(\mathbf{u})$*, since* $\mathbb{A}(s,\mathbf{y}) = \mathbb{B}\{s,\mathbf{F}(\mathbf{y})\}$.

Change point tests X

For change-point problems about the copula, instead of using $\mathbb{B}_n$, Bücher et al. (2014) suggested to use functionals of the sequential empirical process

$$\mathbb{D}_n(s,\mathbf{u}) = \sqrt{n}\lambda_n(s)\{1-\lambda_n(s)\}\left\{\mathcal{C}_n(s,\mathbf{u}) - \tilde{\mathcal{C}}_n(s,\mathbf{u})\right\}, \tag{5}$$

where $\mathcal{C}_n$ and $\tilde{\mathcal{C}}_n$ are sequential empirical copula processes based on the ranks $\{1,\ldots,\lfloor ns\rfloor\}$ of $\mathbf{e}_{n,k}$, $k\in\{1,\ldots,\lfloor ns\rfloor\}$ and the ranks $\{1,\ldots,n-\lfloor ns\rfloor\}$ of $\mathbf{e}_{n,k}$, $k\in\{\lfloor ns\rfloor+1,\ldots,n\}$.
The following theorem extends the results of Bücher et al. (2014) to pseudo-observations.

Theorem 1

Suppose that Assumptions 1 and 2 hold true. Then, as $n\to\infty$, $\mathbb{D}_n \rightsquigarrow \mathbb{D}$ where $\mathbb{D}(s,\mathbf{u}) = \mathbb{B}(s,\mathbf{u}) - \sum_{j=1}^d \mathbb{B}^{(j)}(s,u_j)\partial_{u_j}C(\mathbf{u}) = \mathbb{C}(s,\mathbf{u}) - s\mathbb{C}(1,\mathbf{u})$, $(s,\mathbf{u})\in[0,1]^{1+d}$.

Multipliers I

The next result is an extension of the multiplier central limit theorem results of van der Vaart and Wellner (1996) to empirical processes of pseudo-observations.

The “multipliers” $\xi_1, \ldots, \xi_n$ are iid random variables with mean zero and variance one.

Define the multipliers versions

$$\hat{\alpha}_n(s, \mathbf{y}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \xi_t \left\{ \mathbb{I}\left(\mathbf{e}_{n,t} \leq \mathbf{y} \right) - K_n(\mathbf{y}) \right\},$$

$$\hat{\beta}_n(s, \mathbf{y}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \xi_t \left[\mathbb{I}\left\{ \mathbf{F}_n(\mathbf{e}_{n,t}) \leq \mathbf{u} \right\} - C_n(\mathbf{u}) \right].$$

Multipliers II

The following result shows that the processes

$$\hat{\mathbb{A}}_n(s, \mathbf{y}) = \hat{\alpha}_n(s, \mathbf{y}) - \lambda_n(s)\hat{\alpha}_n(1, \mathbf{y}), \quad (s, \mathbf{y}) \in [0, 1] \times T,$$

and

$$\hat{\mathbb{B}}_n(s, \mathbf{u}) = \hat{\beta}_n(s, \mathbf{u}) - \lambda_n(s)\hat{\beta}_n(1, \mathbf{u}), \quad (s, , \mathbf{u}) \in [0, 1] \times [0, 1]^d,$$

can be used to bootstrap $\mathbb{A}$ and $\mathbb{B}$.

Theorem 2

Suppose that Assumption 1 holds. Then $\hat{\mathbb{A}}_n \rightsquigarrow \hat{\mathbb{A}}$, where $\hat{\mathbb{A}}$ is an independent copy of $\mathbb{A}$.

If Assumption 2 also holds, then $\hat{\mathbb{B}}_n \rightsquigarrow \hat{\mathbb{B}}$, where $\hat{\mathbb{B}}$ is an independent copy of $\mathbb{B}$.

Traditional bootstrap I

Let $\mathbb{A}_n^\star$ and $\mathbb{B}_n^\star$ be the versions of $\mathbb{A}_n$ and $\mathbb{B}_n$ computed with the bootstrap sample of pseudo-observations.

Theorem 3

Suppose that Assumption 1 holds. Then $\mathbb{A}_n^\star \rightsquigarrow \mathbb{A}^\star$, where $\mathbb{A}^\star$ is an independent copy of $\mathbb{A}$.
If Assumption 2 also holds, then $\mathbb{B}_n^\star \rightsquigarrow \mathbb{B}^\star$, where $\mathbb{B}^\star$ is an independent copy of $\mathbb{B}$.

Bootstrapping $\mathbb{D}_n$

In order to bootstrap $\mathbb{D}$, one can use Theorem 1 to get the representation $\mathbb{D}(s, \mathbf{u}) = \mathbb{B}(s, \mathbf{u}) - \sum_{j=1} \mathbb{B}^{(j)}(s, u_j)\partial_{u_j} C(\mathbf{u})$. It then suffices to bootstrap $\mathbb{B}$ according to Theorem 2 or Theorem 3, and estimate the partial derivatives, as in Rémillard and Scaillet (2009).

This bootstrapped version of $\mathbb{D}_n$ is called "nonseq" in the R package npcp (Kojadinovic, 2020). The associated Cramér-von Mises and Kolmogorov-Smirnov statistics are denoted by $\tilde{\mathcal{S}}_n$ and $\tilde{\mathcal{T}}_n$ respectively.

Bücher et al. (2014) also defined another way to bootstrap $\mathbb{D}_n$ by using the two sets of ranks to estimate the partial derivatives of the copula, which is less precise and takes more time to compute. The associated Cramér-von Mises and Kolmogorov-Smirnov statistics are denoted by $\check{\mathcal{S}}_n$ and $\check{\mathcal{T}}_n$ respectively.

Bootstrapping algorithm I

The following algorithm leads to an approximate P-value for $\mathcal{T}_n$ or any continuous functional of $\mathbb{A}_n$, $\mathbb{B}_n$, or $\mathbb{D}_n$. It is stated for the multipliers method but it applies as well to the traditional bootstrap.

After computing $\mathcal{T}_n$ and choosing a large integer N, repeat the following steps for every $k \in \{1, \ldots, N\}$:

1. Generate a random sample $\xi_t^{(k)} \sim N(0,1)$, $t \in \{1, \ldots, n\}$.
2. For $(s, \mathbf{u}) \in [0,1]^{1+d}$, set
   $\hat{\alpha}_n^{(k)}(s, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor ns \rfloor} \xi_t^{(k)} \{\mathbb{I}(\mathbf{e}_{n,t} \leq \mathbf{u}) - K_n(1, \mathbf{u})\}$, and
   $\hat{\mathbb{A}}_n^{(k)}(s, \mathbf{u}) = \hat{\alpha}_n^{(k)}(s, \mathbf{u}) - \lambda_n(s) \hat{\alpha}_n^{(k)}(1, \mathbf{u})$.
3. Evaluate $\mathcal{T}_n^{(k)} = \mathcal{T}_n^{(k)} \left(\hat{\mathbb{A}}_n^{(k)} \right)$.

Bootstrapping algorithm II

An approximate P-value for the test based on $\mathcal{T}_n$ is then given by

$$\frac{1}{N}\sum_{k=1}^{N}\mathbb{I}\left(\mathcal{T}_n^{(k)} > \mathcal{T}_n\right).$$

First experiment I

The first numerical experiment is based on the Dynamic Conditional Correlation (DCC) GARCH model of Engle (2002). The DCC-GARCH model we consider here is specified by the conditional distribution of the bivariate innovations $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \varepsilon_{2t})$, which are normally distributed with mean zero and dynamic covariance structure $\mathbf{q}_t$ given by

$$\mathbf{q}_t = (1 - a - b)\,\boldsymbol{\Omega} + a\,\boldsymbol{\varepsilon}_{t-1}\,\boldsymbol{\varepsilon}_{t-1}^\top + b\,\mathbf{q}_{t-1}, \quad t > 1,$$

where q_1 and $\boldsymbol{\Omega}$ are positive symmetric matrices, $a \geq 0$, $b \geq 0$, and $a + b < 1$. The limiting unconditional covariance matrix of the innovations $\boldsymbol{\varepsilon}_t$ is given by $\boldsymbol{\Omega} = \begin{pmatrix} 25 & 1.5 \\ 1.5 & 1.0 \end{pmatrix}$. Under $\mathcal{H}_0$, we set $a = b = 0$, and under $\mathcal{H}_1$, we set $a = 0.045$ and $b = 0.95$.

First experiment II

Table 1: Percentage of rejection of the null hypothesis for 1000 experiments from the DCC-GARCH model for the Cramér-von Mises statistics $\mathcal{S}_n$, $\tilde{\mathcal{S}}_n$, $\check{\mathcal{S}}_n$ and the Kolmogorov-Smirnov statistics $\mathcal{T}_n$, $\tilde{\mathcal{T}}_n$, $\check{\mathcal{T}}_n$, using the multipliers and bootstrap methods. The theoretical level of the test is 5%. In each case, $N = 100$ bootstrap samples were used to compute the P-values. Since the computation time under H_0 and H_1 were about the same, only the latter is indicated.

	Statistics and bootstrapping method									
	Mult		Boot		Mult		Boot		Mult	
Model	$\mathcal{S}_n$	$\mathcal{T}_n$	$\mathcal{S}_n$	$\mathcal{T}_n$	$\tilde{\mathcal{S}}_n$	$\tilde{\mathcal{T}}_n$	$\tilde{\mathcal{S}}_n$	$\tilde{\mathcal{T}}_n$	$\check{\mathcal{S}}_n$	$\check{\mathcal{T}}_n$
	$n = 250$									
H_0	3.9	4.6	6.0	7.4	6.3	6.6	6.3	6.5	5.3	3.8
H_1	12.7	8.5	12.0	8.4	49.9	39.7	49.9	39.7	51.9	39.1
Seconds	10		16		41		75		6348	
	$n = 500$									
H_0	5.4	5.9	4.2	4.9	4.3	5.4	5.3	4.8	4.9	3.9
H_1	23.0	17.9	24.6	19.2	76.6	65.9	75.2	61.4	76.7	63.6
Seconds	70		114		338		519		51190	
	$n = 1000$									
H_1	40.0	29.0	43.7	35.5	91.3	82.2	91.7	83.2	——	——
Seconds	375		592		2353		3089		——	

First experiment III

- The empirical nominal levels are not significantly different from 5%. Given the computation time, the empirical levels for $n = 1000$ were not computed, as well as the power of $\check{\mathcal{S}}_n$ and $\check{\mathcal{T}}_n$.
- Our results appear to be a bit better than those of Bücher et al. (2014), possibly because of our better estimation of the derivatives of the copula.
- As expected, the power of the three statistics increases as the sample size n increases.
- There is no significant difference in power between the multiplier and the bootstrap methods; however the bootstrap method is much slower.
- For the power, the tests based on the classical sequential process $\mathbb{B}_n$ are clearly outperformed by the those based on the process $\mathbb{D}_n$.

First experiment IV

- There seems to be no significant difference in power between the sequential and non-sequential methods. However, the computation time of the sequential test statistics is prohibitively long, so in practice, it is much better to use the non-sequential method with multipliers.
- The power of the Cramér-von Mises test statistics is always better than the one based on the Kolmogorov-Smirnov test statistic.

Second experiment I

For a given copula in the Clayton, Gumbel, or Gaussian family, for the change-point time $\mathfrak{t} \in \{0.1, 0.25, 0.5, 0.75\}$, and for $n \in \{250, 500\}$, the generating process is the following: the first $\lfloor n\mathfrak{t} \rfloor$ bivariate random vectors $\mathbf{U}_t$ are generated from the copula with a Kendall's tau $\tau_0 = 0.2$, while the remaining values are generated from the copula with Kendall's tau $\tau_1 \in \{0.2, 0.4, 0.6\}$.

$\varepsilon_{tj} = \Phi^{-1}(U_{tj})$, $t \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$, where Φ is the cdf of the standard Gaussian.

These innovations are used to define a GARCH(1,1) sequence for the first component while the second component is a 2-regime Gaussian model

These two series are particular cases of generalized error models satisfying the assumptions. For each sample, we estimated the parameters and we performed change-point tests using the associated pseudo-observations.

Second experiment II

- Even if we used the pseudo-observations from the two generalized error models, the level of the test statistics is not significantly different from the 5% target.
- The results are quite similar for each copula used.
- Also, as in the previous numerical experiments, the test based on $\mathbb{D}_n$ are much more powerful than those based on $\mathbb{B}_n$.

Table 2: Percentage of rejection of the null hypothesis that the copula is the same, using the Cramér-von Mises statistics $\mathcal{S}_n$, $\tilde{\mathcal{S}}_n$, $\check{\mathcal{S}}_n$ and the Kolmogorov-Smirnov statistics $\mathcal{T}_n$, $\tilde{\mathcal{T}}_n$, $\check{\mathcal{T}}_n$ when the distribution of the Clayton copula family changes for the last $\lfloor n\mathfrak{t} \rfloor$ observations. $N = 100$ bootstrap samples are used for the multipliers bootstrap and traditional bootstrap.

Second experiment III

n	τ	$\mathfrak{t}$	Statistics and bootstrapping method									
			Mult		Boot		Mult		Boot		Mult	
			$\mathcal{S}_n$	$\mathcal{T}_n$	$\mathcal{S}_n$	$\mathcal{T}_n$	$\tilde{\mathcal{S}}_n$	$\tilde{\mathcal{T}}_n$	$\tilde{\mathcal{S}}_n$	$\tilde{\mathcal{T}}_n$	$\check{\mathcal{S}}_n$	$\check{\mathcal{T}}_n$
250	0.2	−−	3.6	5.7	4.3	6.4	7.1	5.3	7.2	5.9	7.1	5.1
	0.4	0.10	5.4	6.5	5.5	8.2	19.1	10.7	18.5	10.2	18.2	9.7
		0.25	7.2	6.6	6.9	8.2	48.2	30.1	47.1	28.1	46.3	27.0
		0.50	9.5	8.7	9.0	10.0	56.2	44.9	55.5	45.6	56.9	42.5
		0.75	7.8	9.4	7.4	8.8	25.3	22.1	24.6	20.0	26.2	20.2
	0.6	0.10	5.7	6.8	5.6	7.4	68.4	29.2	67.7	29.1	67.7	24.4
		0.25	12.4	11.3	11.6	12.0	98.8	89.5	98.8	89.2	98.5	85.4
		0.50	22.3	22.5	21.8	24.0	100	98.1	99.9	98.6	99.9	97.8
		0.75	12.4	11.4	12.9	12.0	86.7	75.1	86.6	74.0	86.9	70.4
500	0.2	−−	4.1	6.6	4.4	7.2	6.5	6.5	5.5	6.2	6.3	5.9
	0.4	0.10	5.2	4.9	4.8	6.7	24.1	10.3	21.7	10.7	22.6	8.7
		0.25	7.2	7.8	8.0	9.5	68.0	48.6	68.7	48.7	70.1	46.0
		0.50	13.3	11.8	13.9	11.7	88.5	75.8	87.7	74.8	87.1	73.1
		0.75	9.9	8.4	10.9	8.4	53.6	39.6	52.0	39.9	53.7	37.2
	0.6	0.10	8.0	8.2	8.1	8.3	90.0	50.3	89.9	48.6	89.5	44.5
		0.25	25.0	25.5	25.8	27.5	100	99.5	100	99.6	100	99.6
		0.50	45.6	47.1	44.2	47.8	100	100	100	100	100	100
		0.75	26.7	22.6	25.7	22.7	99.8	98.2	99.8	98.4	99.9	98.6

Change-point for the copula of Nasdaq and S&P 500 indexes (1990-2014) I

- Nasri and Rémillard (2019) modelled the dependence between the "innovations" of daily log-returns of the S&P 500 and Nasdaq indexes from 1990 to 2014 ($n = 6300$ observations), as considered in Adams et al. (2017) where the authors used a DCC-GARCH$(1,1)$.
- We fitted Gaussian HMM models for both series, choosing the number of regimes with goodness-of-fit tests. The Nasdaq returns are modelled by a 6-regime Gaussian HMM, while the S&P 500 returns are modelled by a 5-regime Gaussian HMM (Nasri and Rémillard, 2019).
- In Nasri and Rémillard (2019), using $\mathcal{T}_n$ with multipliers, three break points were found. Here, using $\tilde{\mathcal{S}}_n$ with multipliers, we found five break points, compared to four in Adams et al. (2017).

Change-point for the copula of Nasdaq and S&P 500 indexes (1990-2014) II

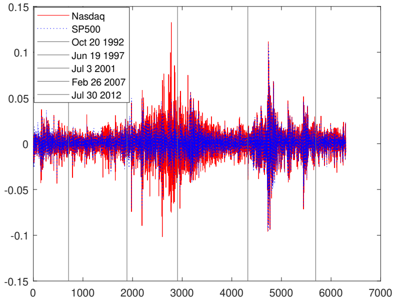

Figure 1: Log-returns for Nasdaq and S&P 500 with change-points on October 20nd 1992, June 19th 1997, July 3rd 2001, February 26th 2007, and July 30th 2012.

Change-point for the copula of the Nasdaq and Dow Jones Industrial indexes (2007–2008) I

- Here, we revisit the second case study in Bücher et al. (2014).
- They did not fit dynamic models to the serially dependent returns. Instead they used dependent multipliers, assuming that the distribution of the log-returns are alpha-mixing.
- The P-value for their test statistic was 59% so there was no evidence of change-points.
- However, the strong mixing hypothesis might not hold. Therefore, we will use the methodology we developed and fit dynamic models to each time series.

Change-point for the copula of the Nasdaq and Dow Jones Industrial indexes (2007–2008) II

- Here, Gaussian HMM models with 3 regimes each seem to be a better fit than GARCH models; in fact univariate change-point tests on the residuals yielded very small P-values compared to the HMM models.
- Based on $\tilde{\mathcal{S}}_n$ (P-value $= 0$ with 1000 multipliers bootstrap samples), there is only one change-point happening on June 5th 2007. This date corresponds to an announcement by Fed chief Bernanke and the indexes started to decrease from their all-time highs (CNN Money).

Change-point for the copula of the Nasdaq and Dow Jones Industrial indexes (2007–2008) III

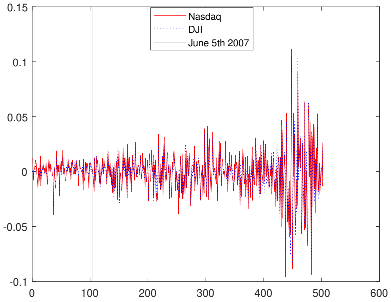

Figure 2: Log-returns for Nasdaq and Dow Jones with change-point on June 5th 2007.

Change-point for the copula of the Nasdaq and Dow Jones Industrial indexes (2007–2008) IV

- Note that the test based on the Kolmogorov-Smirnov statistic had a P-value of 13%. We knew already from the numerical experiments that $\tilde{T}_n$ was not very powerful.
- It is important to point out that using pseudo-observations from fitted dynamic models seem to be more powerful than using dependent multipliers on this raw time series. It is perhaps due to the fact that the series is not strongly mixing.

Conclusion I

- We showed that under weak assumptions, one can used pseudo-observations with change-point tests based on the traditional sequential process and the new process proposed by Bücher et al. (2014) for detecting change-points in the copula.
- We also proposed valid bootstrapping methods to approximate the P-values.
- Based on the numerical experiments, the choice of the bootstrapping method has no significant impact on the power or the level of the tests.
- Most of the time the Cramér-von Mises test outperforms the Kolmogorov-Smirnov test.
- Finally, the tests based on $\mathbb{D}_n$ proposed in Bücher et al. (2014) clearly outperform those based on $\mathbb{B}_n$.

Conclusion II

- For these reasons, the "non-sequential" test statistic $\tilde{\mathcal{S}}_n$ with the multipliers bootstrap should be preferred to any other test considered here since it is the best one in terms of power and it is much faster to compute that the "sequential" test statistic $\check{\mathcal{S}}_n$ which has similar power.
- It seems preferable to work with pseudo-observations instead of using dependent multipliers and assuming mixing conditions.
- Future work:
  - Use weighted processes to increase power and possibly find change-points
  - Consider known change-points in the margins as in Rohmer (2016)
  - Motivated by Berkes et al. (2004), develop sequential change-point methods for early detection problem of infectious diseases, using empirical processes.

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